PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

Volume 8 AUGUST 15, 1922 Number 8

SPACES WITH CORRESPONDING PATHS

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Communicated June 24, 1922

1. In a former paper (these Proceedings, Feb., 1922) Professor Veblen and the writer considered the geometry of a general space from the point of view of the paths in such a space—the paths being a generalization on the geodesics in Riemannian space. It is the purpose of this note to determine the spaces whose paths may be brought into one-to-one correspondence with the paths of a given space, and to consider the degree of arbitrariness involved in our analytical definition of the paths.

2. The equations of the paths in a space S_n are taken in the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0, \tag{2.1}$$

where x^i ($i = 1, \ldots, n$) are the coordinates of a point of a path expressed as functions of a certain parameter s the same for all the paths, and $\Gamma^i_{\alpha\beta}$ are functions of the x's such that $\Gamma^i_{\alpha\beta} = \Gamma^i_{\beta\alpha}$. When the space is Riemannian, with the quadratic form

$$ds^2 = g_{ij}dx^i\,dx^j, \tag{2.2}$$

equations (2.1) define the geodesics of the space, the functions $\Gamma^i_{\alpha\beta}$ being the Christoffel symbols of the second kind formed with respect to (2.2). Thus in choosing (2.1) to define the paths of a general space we have singled out a parameter s which is to be the same for all paths. If we change the independent variable in (2.1), it is found that the resulting equations have the same form only when this parameter is $as + b$, where a and b are arbitrary constants.

3. Suppose we consider a second space S'_n and write the equations of the paths in the form

$$\frac{d^2x^i}{ds'^2} + \Gamma'^i_{\alpha\beta}\frac{dx^\alpha}{ds'}\frac{dx^\beta}{ds'} = 0. \tag{3.1}$$

We say that corresponding points in S_n and S'_n are those for which the coordinates x have the same values in the two spaces. If a path P in S_n is to correspond to a path P' in S'_n, there must be a relation of the form $s = f(s')$ connecting the parameters s and s' at corresponding points. In order that the functions x^i of s' shall satisfy (2.1) and (3.1) we must have

$$\frac{f''}{f'^2} = \frac{\left(\Gamma^i_{\alpha\beta} - \Gamma'^i_{\alpha\beta}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}}{\frac{dx^i}{ds}} = \frac{\left(\Gamma^j_{\alpha\beta} - \Gamma'^j_{\alpha\beta}\right)\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}}{\frac{dx^j}{ds}} \quad (i,j = 1, \ldots, n). \quad (3.2)$$

From these expressions it follows that we must have

$$\left[\left(\Gamma'^i_{\alpha\beta} - \Gamma^i_{\alpha\beta}\right)\frac{dx^j}{ds} - \left(\Gamma'^j_{\alpha\beta} - \Gamma^j_{\alpha\beta}\right)\frac{dx^i}{ds}\right]\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \ (i, j = 1, \ldots, n). \quad (3.3)$$

If it is desired that all the paths of S_n can be brought into one-to-one correspondence with those of S'_n, these equations must be satisfied identically by the quantities $\frac{dx^i}{ds}$, otherwise we should have differential equations of the first order satisfied by all the solutions of (2.1). The conditions that these equations vanish identically are

$$\Gamma'^j_{ii} = \Gamma^j_{ii},\ \Gamma'^i_{jk} = \Gamma^i_{jk} \ (i, j, k \neq), \quad (3.4)$$

and

$$\Gamma'^i_{ii} - 2\Gamma'^j_{ij} = \Gamma^i_{ii} - 2\Gamma^j_{ij},\ \Gamma'^i_{ik} - \Gamma'^j_{jk} = \Gamma^i_{ik} - \Gamma^j_{jk} \ (i, j, k \neq) \quad (3.5)$$

Equations (3.5) are equivalent to

$$\Gamma'^i_{ii} = \Gamma^i_{ii} + 2\varphi_i,\ \Gamma'^i_{ij} = \Gamma^i_{ij} + \varphi_j. \quad (3.6)$$

where the functions φ_i $(i = 1, \ldots, n)$ of the x's are defined by (3.6). By making use of equations (4.2) of the former paper which give the relations between the Γ's in one set of coordinates x and those for another set, we prove that the functions φ_i in (3.6) are the components of a covariant vector.

If we substitute the expressions for $\Gamma'^\alpha_{\beta\gamma}$ from (3.4) and (3.6) in (3.2) we have

$$\frac{f''}{f'^2} = -2\varphi_\alpha \frac{dx^\alpha}{ds}. \quad (3.7)$$

If we substitute in the right-hand member the expressions for the x's as functions of s for a given path, we get the relation between s and s' in order that these functions shall satisfy (3.1) also. Consequently (3.4) and (3.6) are sufficient as well as necessary that the paths of S_n and S'_n be in one-to-one correspondence.

We are in a position now to determine under what conditions the paths of S_n are defined also by equations (3.1). From our definition of paths it follows that s' must be the same for all paths just as s is in (2.1). From (3.7) it is seen that for this to be the case we must have

$$\varphi_\alpha \frac{dx^\alpha}{ds} = \psi(s), \tag{3.8}$$

ψ being the same function for all the paths. Differentiating and making use of (2.1), we obtain

$$\varphi_{\alpha\beta} \frac{dx^\alpha}{ds} \frac{dx^\beta}{ds} = \psi'(s), \tag{3.9}$$

where $\varphi_{\alpha\beta}$ are the components of the covariant derivative of the vector φ_α as defined in the former paper, that is

$$\varphi_{\alpha\beta} = \frac{\partial \varphi_\alpha}{\partial x^\beta} - \varphi_i \Gamma^i_{\alpha\beta}. \tag{3.10}$$

Consider first the case when ψ in (3.8) is constant, that is when (2.1) admits a linear first integral. From (3.9) it follows that S_n must be such that there exists a vector φ_α so that $\varphi_{\alpha\beta} + \varphi_{\beta\alpha} = 0$. When this condition is satisfied by one, or more vectors, the equations of the paths can be written in two, or more ways, in the form (2.1). Spaces which admit a family of parallel covariant vectors, considered by the author in a former paper (these PROCEEDINGS, July, 1922) are of this kind.

If $\psi' =$ const. in (3.9), equations (2.1) would have to admit a quadratic first integral. Another case of this sort arises when $\psi' = a\psi^2 + b$, where a and b are constants. Although these special cases are interesting and lead to types of spaces for which the equations (2.1) are not unique, it is clear that this is not possible for a general S_n. Similar results hold when ψ' is not constant. For on the elimination of s from ψ and ψ' in (3.8) and (3.9) we should obtain a relation between the first derivatives of the x's, which could not hold for all the paths. Hence:

For a general space the equations of the paths can be written in the form (2.1) in only one way, the parameter s being such that it may be replaced by $as + b$, where a and b are arbitrary constants.

4. The components of the curvature tensor in S_n are defined by

$$B^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^\alpha_{jl}\Gamma^i_{\alpha k} - \Gamma^\alpha_{jk}\Gamma^i_{\alpha l} \tag{4.1}$$

If we indicate by primes the functions for S_n', we have from (3.4) and (3.6)

$$\begin{aligned} B'^i_{jkl} &= B^i_{jkl} \ (i, j, k, l \neq), \\ B'^i_{ijk} &= B^i_{ijk} + \varphi_{kj} - \varphi_{jk} \ (i, j, k \neq), \\ B'^i_{jki} &= B^i_{jki} + \varphi_{jk} - \varphi_j \varphi_k \ (i, j, k \neq), \\ B'^i_{jji} &= B^i_{jji} + \varphi_{jj} - \varphi_j^2 \ (i, j \neq), \\ B'^i_{iji} &= B^i_{iji} - \varphi_{ji} + 2\varphi_{ij} - \varphi_i \varphi_j \ (i, j \neq), \end{aligned} \tag{4.2}$$

where $\varphi_{\alpha\beta}$ is the covariant derivative of φ_α. In these formulas there are no summations.

5. By definition the contracted tensor is given by

$$R_{ij} = \sum_\alpha B^\alpha_{ij\alpha} \tag{5.1}$$

Hence from the above formulas we obtain

$$\begin{aligned} R'_{ij} &= R_{ij} + n\varphi_{ij} - \varphi_{ji} - (n-1)\,\varphi_i\,\varphi_j \\ R'_{ii} &= R_{ii} + (n-1)\,(\varphi_{ii} - \varphi_i^2). \end{aligned} \tag{5.2}$$

Since $B^i_{jkl} = -B^i_{jlk}$, a skew symmetric tensor of the second order is defined by

$$S_{ij} = \sum_\alpha B^\alpha_{\alpha ij}. \tag{5.3}$$

Now we have

$$S'_{ij} = S_{ij} + (n+1)\,(\varphi_{ji} - \varphi_{ij}). \tag{5.4}$$

From (5.1), (4.1) and (5.3) we have

$$R_{ij} - R_{ji} = S_{ji}$$

Hence $S_{ij} = 0$ is a necessary and sufficient condition that R_{ij} be symmetric. This result was established by Veblen in a paper presented to the Academy, April 25, 1922.

Veblen (these PROCEEDINGS, July, 1922) has shown that the identity of Bianchi, namely

$$B^i_{jklm} + B^i_{jlmk} + B^i_{jmkl} = 0 \tag{5.5}$$

holds in the geometry of paths; an equivalent result was announced by Schouten (Jahr. Deut. Math. Ver., vol. 30, 1921, Abt., p. 76). In consequence of (5.5) we have from (5.3)

$$S_{ijk} + S_{jki} + S_{kij} = 0, \tag{5.6}$$

or in other form

$$\frac{\partial S_{ij}}{\partial x^k} + \frac{\partial S_{jk}}{\partial x^i} + \frac{\partial S_{ki}}{\partial x^j} = 0. \tag{5.7}$$

We have shown (*Bull. Amer. Math. Soc.*, 1922) that (5.7) is a necessary and sufficient condition that S_{ij} is expressible in the form

$$S_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}, \tag{5.8}$$

where φ_i is a covariant vector, any one component being arbitrary and the others being determined by quadratures. As an immediate consequence we have the theorem:

If S_n is any geometry of paths, spaces S'_n can be found whose paths are in one-to-one correspondence with the paths of S_n and such that the tensor R'_{ij} is symmetric; the determination of S'_n involves an arbitrary function of the x's.

6. For a Riemann space the tensor R_{ij} is symmetric. Consequently

from (5.4) it follows that if S_n and S'_n are Riemann spaces, we must have $\varphi_{ji} = \varphi_{ij}$, that is φ_i must be a gradient.

Suppose now that S_n is any space. From the preceding theorem it follows that we can find a space S_n'' whose paths are in one-to-one correspondence with S_n and such that $S''_{ij} = 0$. If S_n' is a Riemann space in correspondence with S_n, and consequently with S_n'', it follows from (5.4) that the vector giving the relation between S_n'' and S_n' is the gradient of a function φ. The algebraic consistency of equations of the form (cf. (5.3) former paper)

$$g'_{i\alpha} B'^{\alpha}_{jkl} + g'_{\alpha j} B'^{\alpha}_{ikl} =$$

leads to more than $n\,(n + 1)/2$ equations. Hence when the expressions for B'^{α}_{ikl} as given by equations of the form (4.2) are substituted in these equations of condition, we have more than $n(n + 1)/2$ algebraic conditions upon the $n(n + 1)/2$ second derivatives of φ. Consequently the functions B''^{α}_{ijk} for S_n'' are subject to conditions and hence S_n cannot be arbitrary, if its paths are to be in one-to-one correspondence with the geodesics of a Riemann space. It follows also from these remarks that the final theorem of the former paper does not give the complete characterization of geometries whose paths can be identified with the geodesics of a Riemann space.

7. Suppose that S_n is a Riemann space. If S_n' also is to be Riemannian, it follows from (5.4) that φ_i must be the gradient of a function φ. If the components of fundamental quadratic tensors of S_n and S_n' are denoted by g_{ij} and g'_{ij} respectively, we must have

$$\frac{\partial g'_{ij}}{\partial x^k} - g'_{\alpha j}\Gamma'^{\alpha}_{ik} - g'_{i\alpha}\Gamma'^{\alpha}_{jk} = 0,$$

which in consequence of (3.4) and (3.6) becomes

$$\frac{\partial g'_{ij}}{\partial x^k} - g'_{\alpha j}\Gamma^{\alpha}_{ik} - g'_{i\alpha}\Gamma^{\alpha}_{jk} - 2g'_{ij}\frac{\partial \varphi}{\partial x^k} - g'_{kj}\frac{\partial \varphi}{\partial x^i} - g'_{ik}\frac{\partial \varphi}{\partial x^j} = 0. \quad (7.1)$$

If g and g' denote the determinants of g_{ij} and g'_{ij}, we have (Einstein, *Ann. Phys.*, **49**, 1916, p. 796).

$$\frac{1}{\sqrt{g}}\frac{\partial \sqrt{g}}{\partial x^k} = \tfrac{1}{2}\, g^{ij}\frac{\partial g_{ij}}{\partial x^k} = \Gamma^{\alpha}_{\alpha k}. \quad (7.2)$$

Multiplying (7.1) by g'^{ij}, summing for i and j, and making use of (7.2), we obtain an equation from which it follows that

$$\varphi = \frac{C}{2\,(n + 1)}\log\frac{g'}{g}$$

where C is a constant.

Again, if we put $A_{ij} = e^{-4\varphi} g_{ij}$ and indicate by A_{ijk} the covariant derivative of A_{ij} with respect to the fundamental form of S_n, we obtain

$$A_{ijk} + A_{jki} + A_{kij} = 0,$$

which is the condition that the equations of geodesics admit the quadratic integral $A_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds} = \text{const.}$ These results are due to Levi-Civita (*Annali*, **24**, 1896, pp. 273, 276; also *Annalen*, **54**, 1901, p. 188), who has found all the Riemann spaces which admit of geodesic representation. When in particular it is assumed that S_n and S_n' possess corresponding orthogonal systems, and these are taken as parametric we have

$$\Gamma^i_{ii} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x^i},\ \Gamma^i_{ij} = \frac{1}{2g_{ii}}\frac{\partial g_{ii}}{\partial x_j},\ \Gamma^j_{ii} = -\frac{1}{2g_{jj}}\frac{\partial g_{ii}}{\partial x_j},\ \Gamma^k_{ij} = 0,$$

and the equations (3.4) and (3.6) are readily solved; giving

$$g_{ii} = \prod_{j=1}^{n}{}' \left|\psi_j - \psi_i\right|,\ g'_{ii} = \frac{C}{\prod_{j=1}^{n}(\psi_i + c)}\ \frac{g_{ii}}{\psi_i + c},$$

where $j \neq i$ in the first product, ψ_i is an arbitrary function of x^i alone, and c and C are arbitrary constants (Pirro, *Rend. Palermo*, **9**, 1895, p. 169; also, Levi-Civita, l.c., p. 287).

NUMBER OF SUBSTITUTIONS OMITTING AT LEAST ONE LETTER IN A TRANSITIVE GROUP

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Communicated June 10, 1922

In an article published in *Amer. J. Math.*, **26**, 1904, p. 1, H. L. Rietz proved the theorem that in "a primitive group G of degree n of composite order g there are more than $g/(x + 1)$ substitutions of degree less than n, where x is the number of systems of intransitivity of the sub-group which leaves a given letter fixed." If G is a transitive group which is not necessarily primitive this subgroup G_1 is of degree $n - \alpha$, where α is not necessarily unity, and may be supposed to have α transitive constituents of degree 1 in addition to its transitive constituents whose degrees exceed 1. If $x + 1$ represents the total number of these transitive constituents the theorem quoted above remains true for every possible transitive group except the regular groups, as may be inferred from an abstract which appeared in *Bull. Amer. Math. Soc.*, **8**, 1902, p. 17, where the exception noted above is, however, not stated.

The object of the present note is to furnish a very elementary proof of this general theorem, as follows: If each of the substitutions of G_1 involves